INFLUENCE OF VISCOSITY ON THE NATURE OF THE PROPAGATION OF A PLANE EXTENSIONAL SHOCK WAVE

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Shock waves in hereditary elastic media have been studied [1, 2] on the basis of Rabotnov's simplified theory [3] by the small parameter method (the role of the small parameters is taken by the hereditary parameters), which was combined with the method of factorization of nonlinear operators, the theory of discontinuities, asymptotic methods, etc.

In the present article we investigate the propagation of a plane shock wave in nonlinear viscoelastic media characterized by integral and differential viscosity, but without assuming that the viscoelastic parameters are small. Two methods of solution are used: 1) ray tracing [4-9], where the unknown functions after the shock front are represented by power series, whose coefficients are jumps of the corresponding order of derivatives of the displacement; 2) matching of asymptotic solutions [10].

We first consider a nonlinear hereditary elastic medium, i.e., a medium with integral viscosity. The equations describing the motion of such a medium in Eulerian variables in a Cartesian coordinate system have the form

$$\sigma = ku_{,1} + \alpha u_{,1}^2 - \int_0^t K(t-t')\,u_{,1}(t')\,dt' + \ldots,\qquad \sigma_{,1} = \rho_0(u_{,(2)} + 2u_{,1}u_{,1(1)} + \ldots). \tag{1}$$

Here $\sigma = \sigma_{11}$ is the stress; $u = u_1$ is the nonzero component of the displacement vector; $k = \lambda + 2\mu$; λ and μ are the Lamé parameters; $\alpha = 3(\ell + m + n) - (7/2)(\lambda + 2\mu)$; ℓ, m, and n are the third-order elastic constants [11]; $u_{,(k)} = \partial^k u/\partial t^k$; $u_{,k} = \partial^k u/\partial x^k$; t is the time; $x = x_1$ is the coordinate measured along the normal to the boundary of the hereditary elastic half-space $x > 0$; and K(t) is the heredity (elastic memory) kernel. The analogous system of equations in Lagrangian variables has been used [12] to describe the evolution of weak shock waves.

Let the boundary $x = 0$ of the half-space $x > 0$ be loaded, beginning at time $t = 0$, in such a way that $u|_{x=g(t)} = g(t)$ $[g(0) = 0,\ g_{,1}(0) \neq 0,\ g(t) > 0]$. We expand the function g(t) in a Maclaurin series with respect to the time t. Then

$$u(g(t), t) = \sum_{k=1}^{\infty} \frac{1}{k!} n_k t^k \tag{2}$$

(n_k denotes unknown constants). As a result of the dynamic influence of (2), an extensional shock wave propagates in the hereditary elastic medium with velocity G [13]. We formulate the solution after the shock front in the form of a ray series

$$u = -\sum_{k=1}^{\infty} \frac{1}{k!} \varkappa_k \Big|_{t=\int_0^x \frac{ds}{G(s)}} \left(t - \int_0^x \frac{ds}{G(s)}\right)^k, \tag{3}$$

where $\varkappa_k = [u_{c(k)}]$ denotes the jumps of the k-th derivatives of the function u with respect to the time t.

To determine the coefficients $\varkappa_k$ of the ray series (3), we differentiate the first equation of the system (1) k times and the second equation (k − 1) times with respect to t, take their difference on opposite sides of the wave surface Σ, and invoke the compatibility condition [14]

$$[f_{,1(k-1)}] = -G^{-1}[f_{,(k)}] + G^{-1}\frac{\delta [f_{,(k-1)}]}{\delta t}.$$

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As a result, we obtain the recursive equation

$$F_k\left(\varkappa_k, \varkappa_{k-1}, \ldots, \varkappa_1, \frac{\delta\varkappa_{k-1}}{\delta t}, \frac{\delta\varkappa_{k-2}}{\delta t}, \ldots, \frac{\delta\varkappa_1}{\delta t}\right) = 0, \quad k \geqslant 1, \tag{4}$$

which enables us to calculate jumps of any order. From Eq. (4), for k = 1, we obtain the relation $(k + \alpha G^{-1}\varkappa_1 - \rho_0 G^2 - \rho_0 G\varkappa_1 + \ldots)\varkappa_1 = 0$, from which we determine

$$G = c(1 + b\varkappa_1 + \ldots), \tag{5}$$

which coincides with the shock wave velocity in the elastic medium. Here $b = (\alpha - k)k^{-1} \times (2c)^{-1}$, and $c = k^{1/2}\rho_0^{-1/2}$.

Letting k = 2, 3, ... in Eq. (4) and taking the preceding relation into account in each step, we obtain

$$\varkappa_k = f_k\left(\varkappa_1, \frac{\delta\varkappa_1}{\delta t}, \ldots, \frac{\delta^{k-1}\varkappa_1}{\delta t^{k-1}}\right). \tag{6}$$

If we substitute Eqs. (5) and (6) in (3), use the resulting series in Eq. (2), and perform certain algebraic operations, we have

$$\left.\frac{\delta^s\varkappa_1}{\delta t^s}\right|_{t=0} = P_k(n_1, n_2, \ldots, n_{s+1}, \varepsilon, K(0)), \tag{7}$$

where $\varepsilon^2 = n_1 c^{-1}$ and K(0) is the relaxation kernel at t = 0.

The functional relation (7) can be used to represent $\varkappa_1$ by a power series in t or x:

$$\varkappa_1 = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\frac{\delta^k\varkappa_1}{\delta t^k}\right|_{t=0} t^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\left(G^k \frac{d^k\varkappa_1}{dx^k}\right)\right|_{x=0} x^k, \tag{8}$$

and the substitution of Eq. (8) in Eqs. (5) and (6) makes it possible to calculate G and $\varkappa_k$ at times close to the start of the deformation process. If we substitute the values thus obtained for G and $\varkappa_k$ in Eq. (3) and limit the result to the first two terms of the series, after straightforward but tedious calculations we find

$$u = \left(\frac{n_1}{1-\varepsilon^2} - e(\varepsilon)\right)(t-y) + \frac{1}{2b}\left\{-\beta + \left(\frac{7}{2}cb + 1\right) \times \left(-\frac{\varepsilon^2}{1-\varepsilon^2} + \frac{e(\varepsilon)}{c}y\right)\left(\frac{n_1}{1-\varepsilon^2} - e(\varepsilon)y\right)^{-1} e(\varepsilon)\right\}(t-y)^2 + \ldots, \tag{9}$$

where $\beta = K(0)(2\rho_0 c^2)^{-1}$ is the attenuation coefficient for an extensional shock wave in a linear hereditary elastic medium, and

$$y = \left(1 - \frac{bn_1}{1-\varepsilon^2}\right)(be(\varepsilon))^{-1} \ln\left[1 + be(\varepsilon)c^{-1}\left(1 - \frac{bn_1}{1-\varepsilon^2}\right)^{-2} x\right];$$

$$e(\varepsilon) = \left[\frac{n_2}{(1-\varepsilon^2)^3} + \frac{\beta}{b}\right](1-\varepsilon^2)^2 \left[-2\varepsilon^2(1-\varepsilon^2) + \left(\frac{1-\varepsilon^2}{\varepsilon^2} - \frac{7}{2}cb - 1\right)\frac{(1-\varepsilon)^2}{bc}\right]^{-1}.$$

Equation (9) was derived with allowance for the fact that

$$\varkappa_1 = -\frac{n_1}{1-\varepsilon^2} + e(\varepsilon)t, \quad \varkappa_2 = \frac{\beta}{b} + \left[1 + \left(\frac{7}{2}bc + 1\right)\frac{\varkappa_1}{c}\right]\frac{1}{b\varkappa_1}\frac{\delta\varkappa_1}{\delta t}.$$

Relation (9) is simplified somewhat by the natural assumption that ε is small. Then

$$u = n_1[1 - (n_2 b + \beta)y](t-y) + \frac{1}{2b}\left[-\beta + \frac{n_2 b + \beta}{1 - (n_2 b + \beta)y}\right](t-y)^2 + \ldots, \tag{10}$$

where

$$y = \frac{1 - bn_1}{bn_1(n_2 b + \beta)} \ln\left[1 + \frac{bn_1(n_2 b + \beta)}{c(1 - bn_1)^2} x\right].$$

From Eq. (10) with β = 0 we obtain an expression for the displacement in the elastic medium:

$$u = n_1(1 - n_2 by)(t-y) + (1/2)n_2(1 - n_2 by)^{-1}(t-y)^2 + \ldots,$$

$$y = \frac{1 - bn_1}{b^2 n_1 n_2} \ln\left[1 + \frac{b^2 n_1 n_2}{c(1 - bn_1)^2} x\right].$$

Let us analyze the behavior of the jump of $\varkappa_1$, i.e., the coefficient of t − y in Eq. (10), as function of the time. We see that $\varkappa_1$ increases with time for $\beta \le -bn_2$ (b < 0) and decays to zero after a finite time interval $0 \le t \le t^* = (\beta + bn_2)^{-1}$ for $\beta > -bn_2$. In other words, for $\beta > -bn_2$ the shock wave is transformed into a weak shock at time $t = t^*$, despite the active loading of the half-space, and Eq. (10) becomes meaningless at $t > t^*$.

We now consider a nonlinear viscoelastic medium with differential viscosity, the behavior of which is described by the equation

$$\sigma = ku_{,1} + \alpha u_{,1}^2 + \theta v_{,1} + \dots \tag{11}$$

(v is the velocity, and θ is the total viscosity coefficient in shear and bulk deformations).

The foregoing theory of discontinuities can be used to show that a shock wave in the form of a surface of discontinuity cannot propagate in a material whose behavior is described by the model (11), i.e., the ray-tracing method is inapplicable. It will be shown below, however, that a structured shock wave, in which the stress and velocities change rapidly but continuously, can propagate in such a medium.

To formulate a solution, we eliminate the stress σ from Eq. (11) and from the second equation (1) and reduce the result to the dimensionless form

$$\left[1 + 2(\varkappa+1)\left(\varepsilon\frac{\partial w}{\partial s} + \varepsilon^2\frac{\partial w}{\partial m}\right)\right]\left[\frac{\partial^2 w}{\partial s^2} + 2\varepsilon\frac{\partial^2 w}{\partial s\,\partial m} + \varepsilon^2\frac{\partial^2 w}{\partial m^2}\right] - \tag{12}$$
$$-\eta\left(\frac{\partial^3 w}{\partial s^2\,\partial m} + 2\varepsilon\frac{\partial^3 w}{\partial s\,\partial m^2} + \varepsilon^2\frac{\partial^3 w}{\partial m^3}\right)\left(1 + \varepsilon\frac{\partial w}{\partial s} + \varepsilon^2\frac{\partial w}{\partial m}\right) -$$
$$-\eta\varepsilon\frac{\partial w}{\partial m}\left(\frac{\partial^3 w}{\partial s^3} + 3\varepsilon\frac{\partial^3 w}{\partial s^2\,\partial m} + 3\varepsilon^2\frac{\partial^3 w}{\partial s\,\partial m^2} + \varepsilon^3\frac{\partial^3 w}{\partial m^3}\right) - \varepsilon^2\left[\frac{\partial^2 w}{\partial m^2} + 2\varepsilon\frac{\partial w}{\partial m}\left(\frac{\partial^2 w}{\partial s\,\partial m} + \varepsilon^2\frac{\partial^2 w}{\partial m^2}\right)\right] + \dots = 0.$$

Here we have introduced the dimensionless quantities: $m = n_2 n_1^{-1} c^{-1}(x - ct)$, $s = n_2 n_1^{-3/2} \times c^{-1/2} x$, $\varepsilon = n_1^{1/2} c^{-1/2}$, $\eta = n_2 n_1^{-1} k^{-1}\theta$, $\varkappa = 2cb$, $w = n_2 n_1^{-2} u$.

Including two terms in the series, we write the boundary condition (2) in the dimensionless form

$$w - \varepsilon s + m - \frac{1}{2}(\varepsilon s - m)^2\Big|_{s=\varepsilon(\varepsilon s - m) + \frac{1}{2}\varepsilon(\varepsilon s - m)^2} = 0. \tag{13}$$

In the structured shock wave problem the viscosity coefficient η is assumed to be equal to zero everywhere except in a certain thin layer, where η is considered to be small.

We assume that ε is small and seek a solution in the form of an asymptotic series in ε. Since this representation of the solution is valid only near the boundary, we can disregard viscosity in formulating the solution. Accordingly, from Eqs. (12) and (13) we obtain the outer expansion

$$w(s, m) = w^e = f_0 s - m + \frac{1}{2}m^2 + \varepsilon\left[-f_0' s^2 + f_1 s + f_0\left(m - \frac{1}{2}m^2\right)\right] + \tag{14}$$
$$+ \varepsilon^2\left[\frac{2}{3}f_0'' s^3 - f_1' s^2 + f_2 s + \left(\frac{1}{2}m^2 - m\right)(1 - m - f_1)\right] + \dots,$$

in which the functions $f_k(m)$ $(k = 0, 1, 2, \dots)$ are determined from the conditions of matching with the inner expansion, which is valid far from the boundary and decays to zero at infinity.

The formulate the inner expansion, we make the change of variable $n = \varepsilon^k s$ (where k is a certain integer) in Eq. (12). We seek a solution of the transformed equation (12), subject to the extinction condition at infinity, in the series form

$$w(n, m) = w^i = w_0^i + \varepsilon w_1^i + \varepsilon^2 w_2^i + \dots \tag{15}$$

It can be shown that a nonlinear structure shock wave exists only for $k = 3$ [it does not exist for $k = 1$ and for $k > 3$, and it is linear for $k = 2$, i.e., the variables w_i^k $(k = 0, 1, \dots)$ obey linear equations].

If Eq. (15) is inserted into Eq. (12) after its transformation by the substitution $n = \varepsilon^3 s$, the following nonlinear equation must be solved in each step:

$$\frac{\partial^2 w_k^i}{\partial n\,\partial m} + \varkappa\frac{\partial w_k^i}{\partial m}\frac{\partial^2 w_k^i}{\partial m^2} - \nu\frac{\partial^3 w_k^i}{\partial m^3} = \varphi_k^i(n, m, w_0^i, w_1^i, \dots, w_{k-1}^i), \tag{16}$$

which we reduce to an inhomogeneous Burgers equation [15] by the substitution $v_k = \varkappa^{-1}\partial w_k{}^i/\partial m$. Here $\nu = \eta\cdot 2\varepsilon^2$, and the coefficient η is of the order of ε^2.

For the zeroth term of the series (15) the solution of Eq. (16) ($\varphi_0^i \equiv 0$) has the form [15]

$$w_0^i = -\frac{2\nu}{\varkappa}\ln\Phi,\ \Phi = \frac{1}{\sqrt{4\pi\nu n}}\int_{-\infty}^{\infty}\exp\left[-\frac{(m-\zeta)^2}{4\nu n} - \frac{\varkappa}{2\nu}F_0(\zeta)\right]d\zeta \tag{17}$$

[$F_0(\zeta)$ is an unknown function, which is also determined from the matching conditions].

Using the well-known procedure developed by Van Dyke [16] to match the asymptotic expansions (14) and (17), we obtain the equation in dimensioned variables

$$u = \frac{n_2p^2 - \varkappa n_1^2x - 2n_1cp}{2c^2\left(1+\varkappa n_2c^{-2}x\right)} + n_2^2n_1^{-1}\nu\varkappa^{-1}\ln\left(1+\varkappa n_2c^{-2}x\right)$$

($p = x - ct$).

Thus, integral viscosity causes the shock wave to be attenuated, and differential viscosity smears its front, transforming the shock wave into a structured shock wave.

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